The index theorem for the *q*-deformed fuzzy sphere

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Abstract
We calculate the index of the Dirac operator defined on the q-deformed fuzzy sphere. The index of the Dirac operator is related to its net chiral zero modes and thus to the trace of the chirality operator. We show that for the q-deformed fuzzy sphere, a $U_q(su(2))$-invariant trace of the chirality operator gives the q-dimension of the eigenspace of the zero modes of the Dirac operator. We also show that this q-dimension is related to the topological index of the spinorial field as well as to the fuzzy cut-off parameter. We then introduce a q-deformed chirality operator and show that its $U_q(su(2))$-invariant trace gives the topological invariant index of the Dirac operator. We also explain the construction and important role of the trace operation which is invariant under the $U_q(su(2))$, which is the symmetry algebra of the q-deformed fuzzy sphere. We briefly discuss chiral symmetry of the spinorial action on the q-deformed fuzzy sphere and the possible role of this deformed chiral operator in its evaluation using path integral methods.

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1. Introduction

The index of the Dirac operator is known to be related to topological invariance of the underlying space where it is defined. The index of a self-adjoint operator $\mathcal{A}$ is defined as the difference between the number of the zero modes of $\mathcal{A}$ and its adjoint $\mathcal{A}^\dagger$ [1]. Thus, in the case of the Dirac operator, the index is a measure of the net chiral zero modes and thus it is related to the chirality operator Γ and also to the chiral symmetry of the fermionic field theories. The study of index theory of the Dirac operator has been of interest for the understanding of chiral anomaly apart from its importance in understanding the topological features of the underlying space. For fermions coupled to gauge fields, it is known that the index can be expressed in terms of the field strength of gauge fields alone. The index is exactly

the Pontryagin index which is a topological invariant quantity of the underlying space [1]. In the Fujikawa mechanism [2], this topological term is added to the classical action so as to make the partition function invariant under the chiral transformations which is a symmetry of the classical theory [1–3].

The study of the index theorem for Dirac operators on different spaces is of interest not only for the investigation of the topological features but also for the understanding of chiral symmetry. This relation between the zero modes, topological charge of the underlying space and chiral anomaly have been analysed for Dirac operators on various spaces such as two sphere and fuzzy sphere [4–8].

In this paper we obtain the index theorem for a recently constructed [9] Dirac operator on the q-deformed fuzzy sphere. An important feature of this D_q is its invariance under the action of $U_q(su(2))$. Such symmetry plays a determinant role in the computation of the spectrum of D_q. The eigenfunctions of D_q obtained in [9] can be classified into $\pm$ chiral subspaces using the chirality operator Γ. We show a possible modification of the chirality operator obtained in [9] and this deformed chiral operator $\tilde{\Gamma}$ also splits the spinor space into $\pm$ chiral subspace and reduces to chiral operator on fuzzy sphere in the limit $q \to 1$. Using relation of the index of D_q to the trace of the chirality operator, we explicitly find the index of D_q. In obtaining this, we show the important role played by the $U_q(su(2))$ invariant trace. We show that the trace of the chirality operator gives a relation between q-dimension of the eigenspace of the zero modes of D_q and the topological index of the spinor field. We then calculate the index using the deformed chirality operator $\tilde{\Gamma}$. We observe that the indices obtained from chirality operator Γ as well as deformed chirality operator $\tilde{\Gamma}$ have the proper limit with $q \to 1$.

The relation between Hopf algebra (also known as quantum groups) and noncommutative spaces is a well-established matter. In particular, the geometrical and topological aspects of these spaces have been studied in [10, 11]. The construction of field theories on noncommutative geometries with underlying symmetry being $U_q(su(2))$ or its quantum dual $SU(2)_q$ has been explored in [12].

The study of field theory models on noncommutative space in general and on fuzzy S^2 and S^2_{qF} in particular, are of importance as they provide alternate, finite dimensional, regularized models. These models may have advantages such as the absence of fermion doubling problem which plague the usual lattice formulation [13]. Attempts to construct gauge field theories on such manifolds have also been undertaken [14]. We hope that the understanding of the index theory and chiral invariance in these models (à la Fujikawa mechanism) can be helpful in providing us valuable hints in the construction of gauge theories on S^2_{qF}.

The q-deformed fuzzy sphere $S^2_{qF}(N)$ is described by a finite dimensional matrix algebra and carries an action of $U_q(su(2))$. The number N which fixes the dimension of the matrices is called the fuzzy cut-off parameter. In the limit $q \to 1$, this reduces to the usual fuzzy sphere defined as the matrix algebra invariant under $su(2)$.

In other words, $U_q(su(2))$ is the symmetry of the q-deformed fuzzy sphere just as $su(2)$ is the symmetry of the usual fuzzy sphere. We can describe $U_q(su(2))$ as the algebra generated by the operators $J_\pm$ and $K \equiv q^{J_3}$, where q can be a positive real number or a root of unity (i.e., $q = e^{\frac{2\pi i}{p}}$, with p being a positive number) such that the operators satisfy the relations

$$[J_+, J_-] = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad \text{and} \qquad K J_\pm K^{-1} = q^{\pm 1} J_\pm. \tag{1}$$

Here we used the definition of q-number of x

$$[x] \equiv [x]_q = \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{2}$$

$U_q(su(2))$ being a Hopf algebra [15], the Hopf algebra structures, namely, co-product Δ, antipode S and co-unit ϵ of $U_q(su(2))$ are given by

$$\Delta(J_\pm) = J_\pm \otimes K^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes J_\pm, \qquad \Delta(K) = K \otimes K \tag{3}$$

$$S(J_\pm) = -K^{\frac{1}{2}} J_\pm K^{-\frac{1}{2}}, \qquad S(K) = K^{-1} \tag{4}$$

$$\epsilon(J_\pm) = 0, \qquad \epsilon(K) = \epsilon(\mathbb{I}) = 1, \tag{5}$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Observe that the Hopf algebra structures are not preserved by $q \leftrightarrow q^{-1}$. We, thus, have two Hopf algebras $U_q(su(2))$ and $U_{q^{-1}}(su(2))$.

This paper is organized as follows. In section 2, we recall the construction of the spinor module and its chiral decomposition, Dirac operator and its spectrum, on fuzzy sphere. We also show the validity of index theorem for fuzzy sphere Dirac operator here. In section 3, we summarize the essential details of the q-deformed spinor module and its chiral decomposition, Dirac operator and its spectrum for the q-deformed fuzzy sphere which we will be using later. Here we also provide details about the $U_q(su(2))$-invariant trace, which will be crucial for our later discussions. In section 4 we present our main result, namely the derivation of the index theorem for the Dirac operator D_q on S_{qF}^2. Here we show that the trace of the chirality operator Γ relates the q-dimension of the eigenspace of D_q to the topological index of the spinor field and also to the fuzzy cut-off. We then define a q-deformed chirality operator $\tilde{\Gamma}$ and show that its trace gives the topological invariant index of D_q. We conclude in section 5 with comments about the important role of $U_q(su(2))$-invariant trace and its relevance to the chiral anomaly of the theory defined on S_{qF}^2.

2. Dirac operator on the fuzzy sphere

In this section, we briefly recall the essential features of this Dirac operator and its spectrum on fuzzy sphere [16] and show the validity of the index theorem.

The Dirac operator on the fuzzy sphere maps the $su(2)$ spinor module $\left(\mathcal{S}_k^J\right)$ to itself. The spinorial field belonging to this spinor module $\mathcal{S}_k^J$ is defined as

$$\Psi = \Psi^+\left(a_i^\dagger, a_i, b^\dagger, b\right) + \Psi^-\left(a_i^\dagger, a_i, b^\dagger, b\right) = f\left(a_i^\dagger, a_i\right)b + g\left(a_i^\dagger, a_i\right)b^\dagger, \tag{6}$$

where $a_i^\dagger, a_i, i = 1, 2$, are two sets of bosonic creation and annihilation operators, $b^\dagger, b$ is a set of fermionic creation and annihilation operators ($\{b, b^\dagger\} = 1$), and the functions $\Psi^\pm$ can be written as a linear combination of monomials with fixed topological index $2k \in \mathbb{Z}$. Thus we have

$$\Psi^\pm = \sum_{m_1, m_2, \mu, n_1, n_2, \nu} c_{m_1, m_2, \mu, n_1, n_2, \nu} a_1^{\dagger m_1} a_2^{\dagger m_2} b^{\dagger \mu} a_1^{n_1} a_2^{n_2} b^\nu, \tag{7}$$

where m_1, m_2, n_1, n_2 are non-negative integers and $\mu, \nu = 0, 1, \mu + \nu = 1$ satisfying $m_1 + m_2 + \mu \leqslant M, n_1 + n_2 + \nu \leqslant N, M - N = 2k$ apart from the condition $m_1 + m_2 + \mu - n_1 - n_2 - \nu = 2k$. We also have $M + N = 2J$ which is the fuzzy cut-off parameter. Some properties we have to bear in mind concerning these spinorials $\Psi^\pm$ are

- they map the Fock space $\mathcal{F}_N^\mu$ to the Fock space $\mathcal{F}_M^\nu$. These Fock spaces are Hilbert spaces of representations of $su(2)$ and are given by

$$|n_1, n_2; \nu\rangle = \frac{1}{\sqrt{n_1! n_2!}} a_1^{\dagger n_1} a_2^{\dagger n_2} b^{\dagger \nu} |0\rangle, \qquad n_1 + n_2 + \nu = N. \tag{8}$$

- The bosonic parts of the spinor Ψ, i.e., f and g in (6), are tensor representations with even dimensions of $U(su(2))$. Therefore, the angular momentum number l of these operators are half-integers $\frac{2s+1}{2}$, where $s \in \mathbb{Z}$. Further, the bosonic operator f can be written as a $\frac{M}{2} \otimes \frac{(N-1)}{2}$ matrix, and the bosonic operator g can be written as a $\frac{(M-1)}{2} \otimes \frac{N}{2}$ acting on the column vector representation of the Fock space (8). Thus f and g can be written in terms of the tensor operators belonging to the half-integer representations

$$\frac{M}{2} \otimes \frac{N-1}{2} = \left| k + \frac{1}{2} \right| \oplus \cdots \oplus \left(J - \frac{1}{2} \right) \tag{9}$$

and

$$\frac{M-1}{2} \otimes \frac{N}{2} = \left| k - \frac{1}{2} \right| \oplus \cdots \oplus \left(J - \frac{1}{2} \right) \tag{10}$$

respectively where $J = \frac{M+N}{2}$.

- On the spinorial module $\mathcal{S}_k^J$ we define the chirality operator as

$$\Gamma\Psi = -[b^\dagger b, \Psi], \tag{11}$$

which has ± 1 eigenvalues. We denote the respective eigenspaces as $\mathcal{S}_k^{J\pm}$.

- The spinorial module is formed by the linear combination of vectors belonging to the half-integer spin spaces, $\Phi_{J,k+\frac{1}{2},m}^j$ which can be obtained by the repeated action of J_+ (using the co-product) on

$$\Phi_{J,k+\frac{1}{2},-j}^j = \mathcal{N} a_2^{\dagger(j+k+\frac{1}{2})} a_1^{(j-k-\frac{1}{2})}. \tag{12}$$

This spinor module can be splitted into positive chiral and negative chiral subspaces using the above Γ. Thus we have

$$\mathcal{S}_k^J = \mathcal{S}_k^{J+} \oplus \mathcal{S}_k^{J-}. \tag{13}$$

Now the Dirac operator which maps the spinor module to itself and also anti-commutes with Γ is defined using two operators $K_\pm$ and acts on Ψ as

$$D\Psi = K_+\Psi + K_-\Psi. \tag{14}$$

Here $K_\pm$ are operators mapping $\mathcal{S}_k^{J\pm}$ to $\mathcal{S}_k^{J\mp}$. In terms of the operators $a_i^\dagger, a_i, b^\dagger, b$ we write the action of $K_\pm$ as

$$K_+\Psi = ba_2^\dagger \Psi a_1^\dagger b - ba_1^\dagger \Psi a_2^\dagger b \tag{15}$$

$$K_-\Psi = b^\dagger a_1 \Psi a_2 b^\dagger - b^\dagger a_2 \Psi a_1 b^\dagger. \tag{16}$$

We can easily check that with the above definitions, the eigenfunctions of the Dirac operator are $\Psi_{J,k,m}^{j\pm} = \frac{1}{\sqrt{2}}\left[\Phi_{J,k+\frac{1}{2},m} b \pm \Phi_{J,k+\frac{1}{2},m} b^\dagger\right]$ with eigenvalues $\sqrt{\left(j + \frac{1}{2} + k\right)\left(j + \frac{1}{2} - k\right)}$. The $|M - N|$ zero modes are

$$\Psi_{+0}^{m_1 m_2} = \mathcal{N}_1 a_1^{\dagger m_1} a_2^{\dagger m_2} b^\dagger \tag{17}$$

$$\Psi_{-0}^{n_1 n_2} = \mathcal{N}_2 a_1^{n_1} a_2^{n_2} b, \tag{18}$$

where $\mathcal{N}_\alpha, \alpha = 1, 2$, are proportionality constants and allowed value of topological index $2k$ for the above zero modes are $m_1 + m_2 + \mu > 0$ and $n_1 + n_2 + \nu < 0$ respectively and for both these zero modes, the spin $j = |k| - \frac{1}{2}$. Thus it is clear that the number of zero modes are $2j + 1 = 2|k|$ which is equal to $|M - N|$ [16]. It is clear that the trace of Γ restricted to the

space of these zero modes of the Dirac operator will count the net chiral zero modes since each of the $\pm$ chiral zero modes will have ± 1 as eigenvalues under the chiral operator. We have seen that this number is equal to $2k \in \mathbb{Z}$ which is the topological index of the spinor field by construction. Thus we see here that the index theorem is satisfied by the Dirac operator defined above on fuzzy sphere.

D and Γ as operators on the $U(su(2))$-module $\mathcal{S}_k^J$ fulfil the conditions for the index theorem stated above and they are invariant with respect to $su(2)$. The $su(2)$ invariant trace is given by the usual trace on the module $\mathcal{S}_k^J$ and it is this trace we use in evaluating the index of the Dirac operator, D.

3. $U_q(su(2))$ invariant Dirac operator D_q

In this section we present a $U_q(su(2))$ invariant Dirac operator D_q defined on q-deformed fuzzy sphere. The D_q and its spectrum were obtained in [9]. One of the interesting aspects of the spectrum of this Dirac operator is the novel double degeneracy for the case where q is root of unity. Also in this case, we showed that there is a natural cut-off introduced by the root of unity. We now briefly recall the main points of the derivation of D_q which are necessary for the derivation of the index theorem.

Tensor representations of $U_q(su(2))$. A tensor representation of a Hopf algebra is a representation of this algebra on a tensor product of two vector spaces through the use of the co-product. Therefore, in the present case, if $\mathcal{H}_l$ and $\mathcal{H}_l^*$ are a $U_q(su(2))$ irreducible representation and its dual, then the tensor representation on the space $\mathcal{H}_l \otimes \mathcal{H}_l^*$ is given by

$$\rho_\otimes(a)\mathcal{H}_l \otimes \mathcal{H}_l^* \equiv \rho \otimes \bar{\rho}(\Delta(a))\mathcal{H}_l \otimes \mathcal{H}_l^* \subset \mathcal{H}_l \otimes \mathcal{H}_l^* \tag{19}$$

for $a \in U_q(su(2))$ and $\rho : U_q(su(2)) \to \text{Aut}(\mathcal{H}_l)$ is a homomorphism of the algebra $U_q(su(2))$ to the algebra of automorphisms of $\mathcal{H}_l$ and $\bar{\rho} : U_q(su(2)) \to \text{Aut}(\mathcal{H}_l^*)$ is also a homorphism of algebra. If $\mathcal{H}_l$ is the irreducible representation of dimension $2l + 1$, then we see immediately that $\mathcal{H}_l \otimes \mathcal{H}_l^*$ is not irreducible. However, if q is real and positive it can always be decomposed into a direct sum of irreducible representations, i.e.,

$$\mathcal{H}_l \otimes \mathcal{H}_l^* = \bigoplus_{j=0}^{2l} \mathcal{H}_j, \tag{20}$$

where $\mathcal{H}_j$ are the irreducible representations of dimension $2j + 1$. Note that with additional conditions, such a decomposition is possible for q being root of unity also. For $q = e^{\frac{2\pi i}{p}}$, we restrict ourselves to irreducible nilpotent (or classical) representations[3], in the sense that the eigenvalues of $(J_+)^p$, $(J_-)^p$ are both zero, then there is an upper bound for l in equation (20), which is $l_{\max} = \frac{p-1}{2}$.

The tensor representation has for basis the set $\{T_{jk}\}_{j=0,k=-j}^{2l,j}$. We also denote the angular momentum quantum number j as the rank of the tensor. The action of the algebra on these tensors and their eigenvalues are given by

$$\rho_\otimes(J_\pm)T_{jk} = J_\pm T_{jk} K^{-\frac{1}{2}} - K^{-\frac{1}{2}} T_{jk} K^{\frac{1}{2}} J_\pm K^{-\frac{1}{2}} = \sqrt{[j \pm k + 1][j \mp 1]}T_{jk\pm 1},$$

$$\rho_\otimes(K)T_{jk} = K T_{jk} K^{-1} = q^k T_{jk}.$$

Observe that these tensors act as operators on $\mathcal{H}_l$.

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[3] If q is a root of unity there are two types of representations: (i) nilpotent, which can be related to $su(2)$ irreducible representations; (ii) cyclical, which is a new type of representation with no analogue to $su(2)$.

We also define the dual tensor representation. These are representations $\bar{\rho}_\otimes \equiv \bar{\rho} \otimes \bar{\rho}$. We denote the irreducible basis of this dual representation by $T_{jk}^{\ddagger}$. The relation between $T_{jk}^{\ddagger}$ and T_{jk} is given by

$$T_{jk}^{\ddagger} = (-1)^{-k} q^{\frac{k}{2}} T_{j-k}. \tag{21}$$

It is important to note that using these tensor operators with rank $j = \frac{1}{2}$, we can construct the tensors of higher ranks using the q-Clebsch–Gordan coefficients [15] (Therefore it is enough to show the results explicitly for the q-tensor operators with $j = \frac{1}{2}$).

Let us denote by $A_i^{\dagger}$, A_i, with $i = 1, 2$, two sets of q-bosonic creation and annihilation operators. These operators satisfy the following relations:

$$A_i A_i^{\dagger} - q^{\frac{1}{2}} A_i^{\dagger} A_i = q^{\frac{-N_i}{2}} \tag{22}$$

$$\left[N_i, A_i^{\dagger}\right] = A_i^{\dagger} \tag{23}$$

$$[N_i, A_i] = -A_i, \tag{24}$$

where $A_i^{\dagger} A_i = [N_i]$ and N_i is the number operator. The q-bosonic operators and the number operator satisfying these relations are known as q-Heisenberg algebra.

We can write the tensor operators with rank $j = \frac{1}{2}$ [15] in terms of these q-bosonic operators

$$\alpha_1^{\dagger} \equiv T_{\frac{1}{2}\frac{1}{2}} = A_1^{\dagger} q^{-\frac{N_2}{4}} \tag{25}$$

$$\alpha_2^{\dagger} \equiv T_{\frac{1}{2}-\frac{1}{2}} = q^{\frac{N_1}{4}} A_2^{\dagger} \tag{26}$$

and the dual representation (21) of these tensors as

$$\alpha_1 \equiv T_{\frac{1}{2}-\frac{1}{2}}^{\ddagger} = A_1 q^{-\frac{N_2+1}{4}} \tag{27}$$

$$\alpha_2 \equiv T_{\frac{1}{2}\frac{1}{2}}^{\ddagger} = -q^{\frac{N_1+1}{4}} A_2. \tag{28}$$

The set of operators $\alpha_i^{\dagger}$, α_i satisfies

$$\alpha_1^{\dagger} \alpha_2^{\dagger} = q^{-\frac{1}{2}} \alpha_2^{\dagger} \alpha_1^{\dagger} \tag{29}$$

$$\alpha_1 \alpha_2 = q^{\frac{1}{2}} \alpha_2 \alpha_1 \tag{30}$$

$$\alpha_1^{\dagger} \alpha_2 = \alpha_2 \alpha_1^{\dagger} \tag{31}$$

$$\alpha_1 \alpha_2^{\dagger} = \alpha_2^{\dagger} \alpha_1. \tag{32}$$

Spinor module. The spinor module $\mathcal{S}$ is made of half-integer angular momentum tensor operators combined with a set of fermionic creation and annihilation operators $b^{\dagger}$, b. This set of fermionic operators is the usual ones, i.e., $\{b, b^{\dagger}\} = 1$. A basis of this spinorial module is a monomial of the form

$$\alpha_1^{\dagger m_1} \alpha_2^{\dagger m_2} b^{\dagger \mu} \alpha_1^{n_1} \alpha_1^{n_2} b^{\nu} \propto A_1^{\dagger m_1} A_2^{\dagger m_2} b^{\dagger \mu} A_1^{n_1} A_1^{n_2} b^{\nu} \tag{33}$$

where the proportionality constant would involve q and N_α. In the above m_1, m_2, n_1, n_2 are non-negative integers and $\mu = \nu = 0, 1$, $\mu + \nu = 1$. Also $m_1 + m_2 + \mu \leqslant M$ and

$n_1 + n_2 + \nu \leqslant N$, $M - N = 2k$, $m_1 + m_2 + \mu - n_1 - n_2 - \nu = 2k$. We also have $M + N = 2J$ which act as the fuzzy cut-off parameter. Similar to the previous section, we define the spinorial fields as

$$\Psi = \Psi^+\left(A_i^\dagger, A_i, b^\dagger, b\right) + \Psi^-\left(A_i^\dagger, A_i, b^\dagger, b\right) = f\left(A_i^\dagger, A_i\right)b + g\left(A_i^\dagger, A_i\right)b^\dagger, \tag{34}$$

such that

$$\Psi^\pm = \sum_{m_1,m_2,\mu,n_1,n_2,\nu} c_{m_1,m_2,\mu,n_1,n_2,\nu} A_1^{\dagger m_1} A_2^{\dagger m_2} b^{\dagger \mu} A_1^{n_1} A_2^{n_2} b^\nu, \tag{35}$$

where c_{m_1,m_2,n_1,n_2} are complex numbers multiplied by factors involving q and N_α. The bosonic part of Ψ in equation (34), i.e., f and g can be expressed in terms of the spin-half tensors belonging to $\frac{M}{2} \otimes \frac{(N-1)}{2}$ and $\frac{(M-1)}{2} \otimes \frac{N}{2}$ respectively and can be decomposed into the direct sum of IRR of half-integer spins as shown in equations (9) and (10). The spinorial fields Ψ are built by taking the linear combinations of the operators belonging to these spin-half spaces $\Phi^j_{J,k+\frac{1}{2},m}$, where the lowest weight state is given by

$$\Phi^j_{J,k+\frac{1}{2},-j} = \left(A_2^\dagger q^{\frac{N_1}{4}}\right)^{(j+k)}\left(A_1 q^{-\frac{N_2+1}{4}}\right)^{(j-k)}. \tag{36}$$

q-Deformed Dirac operator.   The chirality operator is defined as in the usual $su(2)$ case

$$\Gamma_q \Psi \equiv \Gamma \Psi = -[b^\dagger b, \Psi], \tag{37}$$

where the subscript q is used to distinguish it from the chirality operator on fuzzy sphere and for comparison of results in the S^2_{qF} case to the usual fuzzy one. This chirality operator splits the $U_q(su(2))$ spinor module into $\pm$-chiral subspaces.

As in the usual $su(2)$ case we construct auxiliary operators $K_\pm$, which map $\pm$ chiral subspace to $\mp$ chiral subspace. We require also that these operators be invariant with respect to $U_q(su(2))$. In [9], it was shown that the operators that fulfil these conditions are

$$K_+ \Psi = q^{-\frac{k-m}{4}} K^{-\frac{1}{2}} b\left[A_1^\dagger \Psi A_2^\dagger q^{\frac{k}{2}} - A_2^\dagger \Psi A_1^\dagger\right]b \tag{38}$$

$$K_- \Psi = q^{-\frac{k-m}{4}} b^\dagger\left[A_1 \Psi A_2 q^{\frac{k}{2}} - A_2 \Psi A_1\right]b^\dagger K^{-\frac{1}{2}}. \tag{39}$$

The q-deformed fuzzy Dirac operator D_q is defined by

$$D_q \Psi = K_+ \Psi + K_- \Psi \tag{40}$$

and thus D_q anti-commutes with the chirality operator Γ_q. It is easy to see that the zero modes of this q-Dirac operator are given by [9]

$$\Psi_{0+}^{m_1 m_2} = \mathcal{N}_1 A_1^{\dagger m_1} A_2^{\dagger m_2} b^\dagger \tag{41}$$

$$\Psi_{0-}^{n_1 n_2} = \mathcal{N}_2 A_1^{n_1} A_2^{n_2} b, \tag{42}$$

where $\mathcal{N}_\alpha$, with $\alpha = 1, 2$, are normalization constants and they involve factors of q and the number operators N_α. These zero modes correspond to the angular momentum number $j = |k| - \frac{1}{2}$. The number of these zero modes is given by $|M - N| = 2j + 1 = 2|k|$. The non-zero eigenvalues of this Dirac operator are $\sqrt{\left[j + \frac{1}{2} + k\right]\left[j + \frac{1}{2} - k\right]}$ and corresponding eigenfunctions are

$$\Psi_{J,k,m}^{j\pm} = \frac{1}{\sqrt{2}}\left[\Phi_{J,k+\frac{1}{2},m} b \pm \Phi_{J,k+\frac{1}{2},m} b^\dagger\right], \tag{43}$$

where $\Phi_{J,k\pm\frac{1}{2},m}$ is obtained from equation (36). Here we note that the eigenvalues are real. This is true even for the case of $q = \mathrm{e}^{\frac{2\pi\mathrm{i}}{p}}$ provided $2l + 1 < p$ which is a constraint one naturally introduce on the fuzzy cut-off parameter [9]. This condition is also needed to ensure the positivity of the inner product as explained in the next section. The reality of the eigenvalues clearly shows that our Dirac operator is self-adjoint as required. We also note that the chirality operator Γ anti-commutes with the Dirac operator and it splits the spinor module into $\pm$-chiral subspaces.

4. Index theorem for D_q

In the previous section we have presented the $U_q(su(2))$ invariant Dirac operator which anti-commutes with a chiral operator $\Gamma_q \equiv \Gamma$ and we have given its zero modes [9]. Both D_q and Γ_q act on a $U_q(su(2))$ spinorial module $\mathcal{S}$. In order to apply the index theorem to these operators we have to first define a $U_q(su(2))$-invariant trace which is also essential in showing that D_q is self-adjoint.

$U_q(su(2))$-invariant trace. A trace acting on operators of a Hopf algebra module V is said to be invariant if it satisfies

$$\mathrm{Tr}(\rho_\otimes(\Delta(a))\hat{T}) = \epsilon(a)\,\mathrm{Tr}(\hat{T}), \tag{44}$$

where $\epsilon(a)$ is the co-unit of the element a of the Hopf algebra and $\hat{T}$ is an element of vector space $V \otimes V^*$.

We can easily see that since the co-product of $U_q(su(2))$ is deformed, the above condition for the usual trace on the $U_q(su(2))$-module $\mathcal{H}_l$ is not satisfied. In order to have an invariant trace on a representation $\mathcal{H}_l$, we need to deform the trace as well. The general recipe to construct this deformed trace, which is called q-trace here, is given in [17]. There it is shown that the usual trace is a linear functional on the space $\mathcal{H}_l \otimes \mathcal{H}_l^*$. However, in general, for quasi-triangular Hopf algebras this trace is not invariant. So a new trace is defined to be a linear functional on the space $\mathcal{H}_l^{**} \otimes \mathcal{H}_l^*$. It can be shown that $\mathcal{H}_l^{**} \simeq \mathcal{H}_l$ as a representation for these quasi-triangular Hopf algebras.

In the present case of $U_q(su(2))$, the equivalent map $\mathcal{H}_l \otimes \mathcal{H}_l^*$ to $\mathcal{H}_l^{**} \otimes \mathcal{H}_l^*$ is given by multiplication by the matrix representation of the generator $K = q^{J_3}$, i.e., the tensor basis $T_{lm} \in \mathcal{H}_l \otimes \mathcal{H}_l^*$ goes to $K \cdot T_{lm} \in \mathcal{H}_l^{**} \otimes \mathcal{H}_l^*$. Thus, we define the q-trace, denoted by Tr_q, as

$$\mathrm{Tr}_q(T_{lm}) = \mathrm{Tr}(KT_{lm}) \tag{45}$$

where on the right-hand side we have the usual matrix trace [7]. Now, one can easily check the invariance of this trace with respect to $U_q(su(2))$. Indeed

$$\mathrm{Tr}(\bar{\bar{\rho}} \otimes \bar{\rho}(\Delta(a))(KT_{lm})) = \epsilon(a)\,\mathrm{Tr}(KT_{lm}).$$

Observe that we have to consider the proper representation with double bar in the first term of the tensor product. We also note that $\bar{\bar{\rho}}(a) = \rho(S^2(a))$, S being the antipode.

In the space $\mathcal{H}_l \otimes \mathcal{H}_l^*$ we can define a positive definite inner product using the q-trace as

$$(A, B) = \mathrm{Tr}_q(A^\ddagger B), \tag{46}$$

where $A, B \in \mathcal{H}_l \otimes \mathcal{H}_l^*$ and $A^\ddagger$ is in the dual of $\mathcal{H}_l \otimes \mathcal{H}_l^*$.

It can be easily seen that the reality condition $\overline{(A, B)} = (B, A)$ is satisfied for the case of q being real. But for q being root of unity, it is satisfied only if we define the inner product with

$$A^\circledast = K^{-1}A^\ddagger, \tag{47}$$

such that $A^{\circledast\circledast} \neq A$ and

$$(A, B) = \mathrm{Tr}_q(A^{\circledast} B) = \mathrm{Tr}(A^{\ddagger} B), \tag{48}$$

where on the right-hand side Tr is the usual trace. Thus we see that the reality condition is satisfied for q being root of unity also. Further we observe that the positivity of this inner product is only satisfied if $2l + 1 < p$, where $q = \mathrm{e}^{\frac{2\pi i}{p}}$.

Thus we have now defined $U_q(su(2))$-invariant trace which allows one to define the self-adjoint of a given operator and are in a position to evaluate the trace of the chirality operator on S^2_{qF} and obtain the index theorem.

Index theorem for D_q. The $U_q(su(2))$ invariant self-adjoint Dirac operator D_q and the chiral operator Γ_q on q-deformed fuzzy sphere do satisfy all the requirements listed for a generic self-adjoint operator in section 2. We have also presented a q-trace (45) and the definition of self-adjointness with respect to this new trace. Now, we are in position to apply the index theorem to the operator D_q. We first calculate the q-trace of Γ and then the q-trace of a deformed operator $\tilde{\Gamma}$, which we will define.

(1) *Invariant trace of chiral operator (Γ_q) and index.* First we consider chirality operator defined on the q-deformed fuzzy sphere given in equation (37) to derive the index. Using the definition of the invariant trace and zero modes we get

$$\mathrm{Tr}_{q,D=0}(\Gamma_q) = [n_+] - [n_-]. \tag{49}$$

This can be re-expressed using identities involving the q-numbers [15] as

$$[n_+] - [n_-] = \frac{[n_+ + n_-]\left[\frac{(n_+ - n_-)}{2}\right]}{\left[\frac{(n_+ + n_-)}{2}\right]}. \tag{50}$$

Since $n_+ - n_-$ is the number of net chiral zero modes of D_q which is equal to $2k$ and $n_+ + n_- = 2J$ ($2J = M + N$ is the fuzzy cut-off parameter), we get

$$\mathrm{Tr}_{q,D=0}(\Gamma_q) = \frac{[2J][k]}{[J]}. \tag{51}$$

Observe that when $q \rightarrow 1$, we get $\mathrm{Tr}(\Gamma) = 2k$ which is the correct result in this limit. Thus we observe that unlike in the case of usual fuzzy sphere, here the trace of Γ_q gives q-number of the topological charge multiplied by factors which depend on the sum of $n_{\pm}$. Thus we note that the trace of Γ_q does not give just the count of net chiral zero modes in the case of S^2_{qF}.

The q-trace of the identity operator on a vector space is also known as the q-dimension or quantum dimension [17, 18] of this vector space. This quantity plays an important role in the representation theory of quantum groups [18]. Therefore, we can see the left-hand side of equation (51) as the difference of the q-dimensions of the subspaces of positive and negative chiralities of the spinor module $\mathcal{S}$, respectively. The right-hand side is a quantity depending on the topological number k of the spinorial field. We can thus consider a generalization of the index theorem for the Dirac operator on the q-deformed sphere as

$$q\,\mathrm{Index}\,D \equiv q\,\dim(\mathrm{Ker}\,D) - q\,\dim(\mathrm{Ker}\,D^{\dagger}), \tag{52}$$

where $q\,\dim(\mathrm{Ker})$ is the q-dimension of the space of zero modes of the respective operators. Then, equation (51) can be written as

$$q\,\mathrm{Index}\,D = \frac{[2J][k]}{[J]}. \tag{53}$$

Here we note that the q-index depends on the fuzzy cut-off parameter $2J = M + N$. This novel feature of the q-deformed fuzzy sphere is absent in the usual fuzzy sphere where the fuzzy cut-off is not known to have any effect on the topological properties.

(2) *Invariant trace of deformed chiral operator* ($\tilde{\Gamma}$) *and index*. Now we consider a deformed chirality operator $\tilde{\Gamma} = K^{-1}\Gamma$. Since $DK = KD$ we have $DK^{-1}\Gamma + K^{-1}\Gamma D = 0$. Though this deformed chiral operator $K^{-1}\Gamma$ is not an involution ($(K^{-1}\Gamma)^2 \neq 1$), we note that it also splits the spinor module into $\pm$ chiral subspaces and in the limit $q \to 1$, it correctly reduces to the chirality operator on S_F^2 as required. Due to the K^{-1} factor, when acted on the chiral spinors, it picks up an extra q^{-m} factor, apart from the ± 1. This deformed chirality operator acts on a spinor as $\tilde{\Gamma}\Psi = -\Delta(q^{-J_3})[b^\dagger b, \Psi]$. For instance, $\tilde{\Gamma}\Psi_{J,k,m}^{j\pm} = \pm q^{-m}\Psi_{J,k,m}^{j\pm}$. Thus $(\tilde{\Gamma})^2$ acting on $\Psi_{J,k,m}^{j\pm}$ gives $q^{-2m}\Psi_{J,k,m}^{j\pm}$. This still split the spinor module into $\pm$ chiral subspaces. With this deformed chirality operator we find that

$$\text{Tr}_{q,D=0}(K^{-1}\Gamma) = n_+ - n_- = 2k. \tag{54}$$

Thus we see that the trace of the deformed chiral operator does give the net number of chiral zero modes as in the usual case. We also know that by construction [9] this difference in the number of $\pm$ chiral zero modes is equal to the topological index $2k$ of the spinorial field. Since here the trace has no dependence on fuzzy cut-off or q, we are guaranteed to get the correct result in the limit of S_F^2 and also in the limit of continuum sphere.

Thus we see that the trace of chirality operator Γ as well as that of deformed chiral operator $\tilde{\Gamma}$ do reproduce the expected result in the limit of $q \to 1$. But for generic q, it is $\tilde{\Gamma}$ which gives the count of net chiral zero modes. Thus it seems that the deformed chiral operator may be more natural and of more use in the study of chiral anomaly on deformed fuzzy sphere.

5. Conclusion

In this paper we have obtained the index of the Dirac operator defined on q-deformed fuzzy sphere S_{qF}^2. Since the index is a measure of the net chiral zero modes, we calculate it by evaluating the trace of the chirality operator defined on S_{qF}^2. For this calculation we have used the $U_q(su(2))$ invariant trace [7, 17]. We have shown that the trace of chiral operator Γ is proportional to the $[n_+ - n_-]_q$. The proportionality constant depends on the q-number of the fuzzy cut-off parameter. This situation is strikingly different from the undeformed fuzzy sphere where the cut-off parameter does not affect the topological invariants of the manifold. Using the explicit form of zero modes of the Dirac operator D_q on S_{qF}^2, we exhibit that $n_+ - n_- = 2k$, where $2k \in \mathbb{Z}$ is the topological index of the spinor field. Thus, with this chirality operator, we get the index of Dirac operator proportional to the q-number of the topological index. Since the $U_q(su(2))$-invariant trace of the chirality operator is the q-dimension of the eigenspace of D_q, this shows that the q-dimension of the eigenspace is related to the topological index of the spinor field. Since the q-number reduces to usual number in the limit $q \to 1$, we see that $\left[\frac{n_+ - n_-}{2}\right]_q \to k$ in this limit as required. Here we have seen that the proportionality constant depends on the sum of total chiral zero modes and in the limit this proportionality constant becomes unity. Then we have shown that a deformed chirality operator can be used in place of Γ and its $U_q(su(2))$-invariant trace is just $2k$ as in the usual case. Though this deformed chiral operator $\tilde{\Gamma}$ is not an involutive operator, it does split the spinor module into $\pm$ chiral subspace and also reduces to the correct chiral operator in the $q \to 1$ limit.

Using the $U_q(su(2))$-invariant trace, we can construct the invariant spinorial action on q-deformed fuzzy sphere S_{qF}^2. For this, we can use q-Clebsch–Gordan techniques and combine Ψ, $D_q\Psi$ and their duals defined using the invariant trace. Therefore terms such as $\Psi^\ddagger\Psi$ and

$\Psi^\ddagger D_q \Psi$ are invariants with respect to $U_q(su(2))$ (For showing these we use equation (21)). Thus the $U_q(su(2))$ invariant spinorial action is finally given by

$$S = \frac{2\pi R^2}{[N+1]} \mathrm{Tr}_q(\bar{\Psi} D_q \Psi + V(\bar{\Psi}\Psi)),$$ (55)

where, $\bar{\Psi} = \Psi^\ddagger$, see (46), or $\bar{\Psi} = \Psi^\circledast$, see (48), depending on whether q is real or root of unity, $[N+1]$ is the normalization factor, R is the radius of the underlying sphere and $V(\bar{\Psi}\Psi)$ is a potential function on the $U_q(su(2))$ invariant $\bar{\Psi}\Psi$. It can be easily seen that the above action is also invariant under chiral transformations for the massless case. Now, one may ask if the chiral symmetry of the action is still a symmetry of the corresponding quantum theory.

Consider the chiral invariant action $\mathcal{S} = S|_{V=0}$ where the action S is given in equation (55). The corresponding partition function is

$$\mathcal{Z} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, e^{-\mathcal{S}}.$$ (56)

It is not invariant with respect to the chiral transformations[4]

$$\Gamma : \Psi \to e^{i\alpha\Gamma}\Psi, \qquad \Gamma : \bar{\Psi} \to \bar{\Psi} \, e^{i\alpha\Gamma},$$ (57)

where α is a real parameter. As in the case of spinorial action in the commutative space we note that the contributions from $\pm$ chiral non-zero modes to the Jacobian of chiral transformation cancel each other. But the contribution coming from the zero modes does add up to give

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \to e^{|n_+ - n_-|}\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$
$$= e^{2k}\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$ (58)

which breaks the chiral symmetry of the quantum theory. We note here that the above contribution to the integration measure is the same for the chiral transformations generated by Γ as well as $\tilde{\Gamma}$. Thus we see that by defining an effective action with a counter-term that cancels the above contribution, one can cancel the anomaly and retain the chiral invariance. Since this new term has to be invariant under $U_q(su(2))$, we need to define this term using $U_q(su(2))$-invariant trace. Also since the extra phase factor is just $2k$ and not proportional to $[k]_q$ we see that the effective action depends on the deformed chirality operator $\tilde{\Gamma}$ rather than the chirality operator Γ_q. Thus we get the effective action to be

$$\mathcal{S}_{\mathrm{eff}} = \mathcal{S} - \mathrm{Tr}_q(\bar{\Psi} K^{-1} \Gamma \Psi).$$ (59)

Therefore, the deformed chirality operator may be better suited for the study of spinor fields and their actions on S^2_{qF}.

In the commutative spaces, the new term in the effective action in equation (59) has been expressed in terms of the gauge fields alone [1]. There have been some studies aiming in the construction of gauge field theories on fuzzy sphere [14]. The path integral evaluation of above term can lead to better understanding of the construction of gauge field action on q-deformed fuzzy sphere as well as on fuzzy sphere.

Acknowledgments

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[4] The chiral transformations effected by the deformed chiral operator $\tilde{\Gamma} = q^{-J_3}\Gamma$ is $\tilde{\Gamma}\cdot\Psi = \Delta(q^{-J_3}) \, e^{i\alpha\Gamma}(\Psi)$ where $\Delta(J_3)$ acts on the bosonic part, i.e., f and g of Ψ and Γ acts on the fermionic part as in equation (57).

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